

Reference angle theorem

A reference angle for an angle θ in standard position is the positive acute angle formed by the x -axis and the terminal side of θ . Because reference angles are always positive and acute, that means reference angles will always measure between 0° and 90° , or between 0 and $\pi/2$ radians.

Which means that when

θ is in quadrant I, θ 's reference angle β will be $\beta = \theta$.

θ is in quadrant II, θ 's reference angle β will be $\beta = \pi - \theta$.

θ is in quadrant III, θ 's reference angle β will be $\beta = \theta - \pi$.

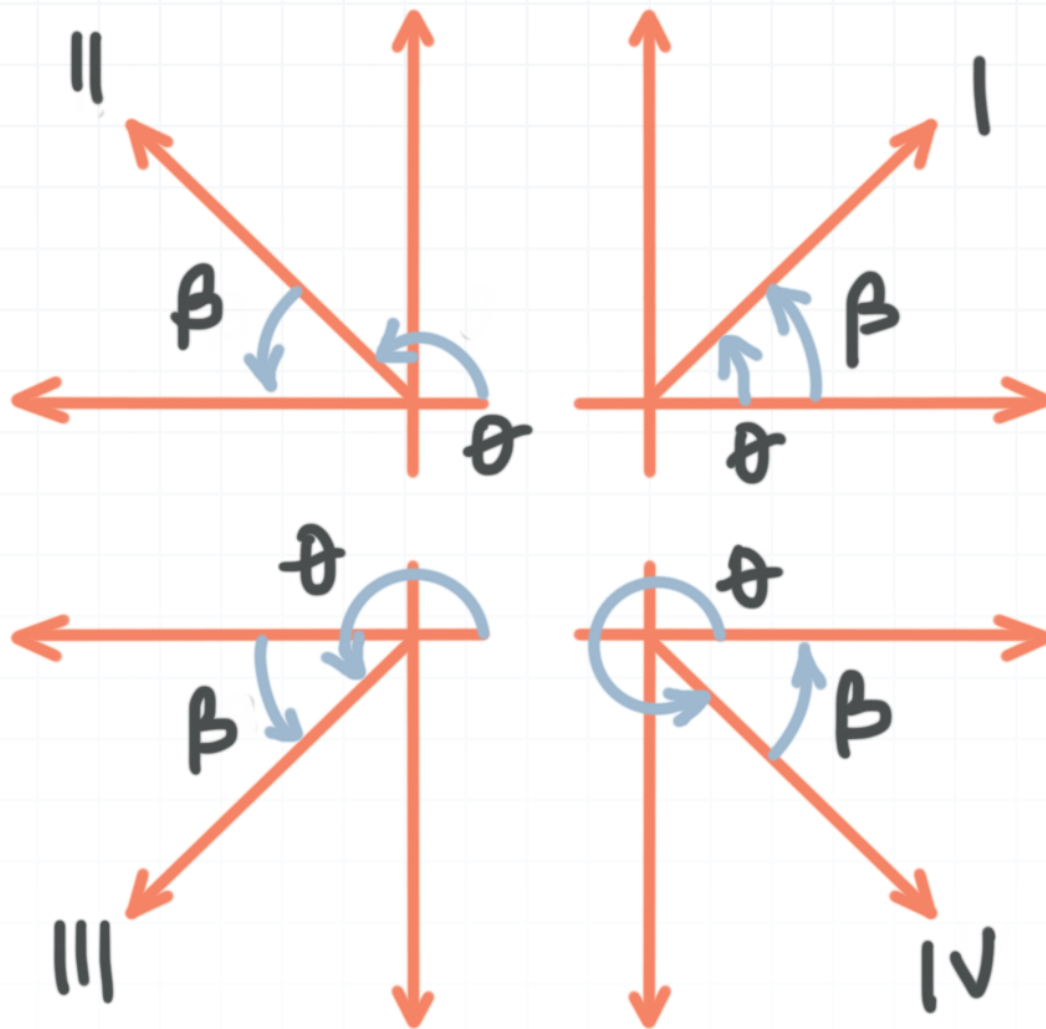
θ is in quadrant IV, θ 's reference angle β will be $\beta = 2\pi - \theta$.

Let's make a table that includes the reference angle for both radians and degrees.

θ 's quadrant	β in rad	β in degrees
I	$\beta = \theta$	$\beta = \theta$
II	$\beta = \pi - \theta$	$\beta = 180^\circ - \theta$
III	$\beta = \theta - \pi$	$\beta = \theta - 180^\circ$
IV	$\beta = 2\pi - \theta$	$\beta = 360^\circ - \theta$

If β is the reference angle for θ , we can sketch examples of θ and β in each quadrant.





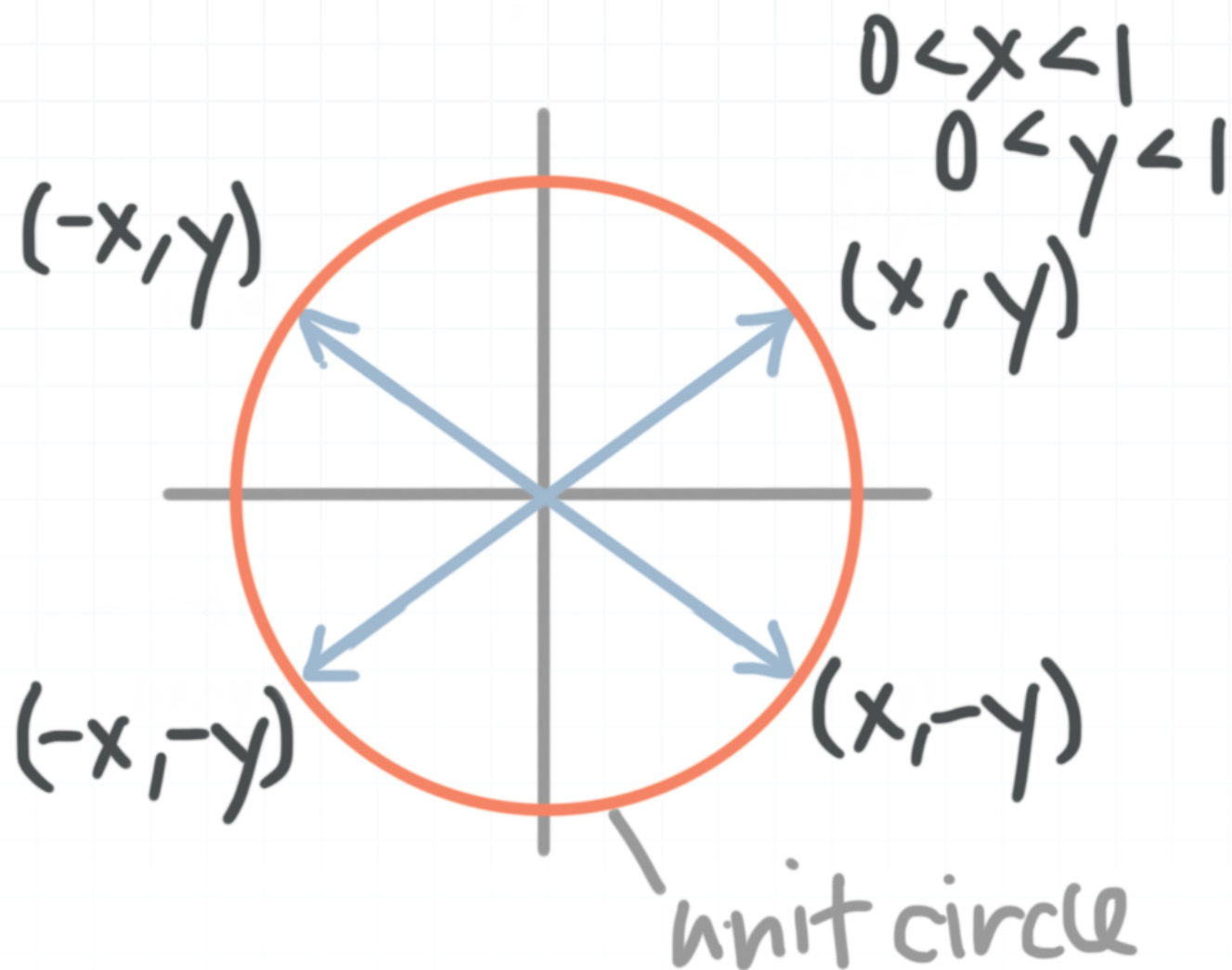
Notice that all of these angles θ are in the positive direction. In order to use the equations in the table above to find the reference angle, we need θ to be positive. If we have an angle θ that's negative, then we need to find the positive coterminal angle and use that instead.

We can also take advantage of angle symmetry across both the x - and y -axis. When we reflect an angle across the y -axis, we know that the value of sine will be the same, since the sine of an angle gives you the y -coordinate. And when we reflect an angle across the x -axis, we know that the value of cosine will be the same, since the cosine of an angle gives you the x -coordinate.

Putting all this together tells us that we really only need to know the sine and cosine values (y - and x -coordinates, respectively) of angles in the first



quadrant, and from symmetry and reference angles we can find the coordinates of all the angles in the other three quadrants.



Example

What is the reference angle for $\theta = 2\pi/3$?

The angle $\theta = 2\pi/3$ is in the second quadrant, which means the reference angle β is

$$\beta = \pi - \theta$$

$$\beta = \pi - \frac{2\pi}{3}$$



$$\beta = \frac{3\pi}{3} - \frac{2\pi}{3}$$

$$\beta = \frac{\pi}{3}$$

Let's do an example with an angle given in degrees.

Example

What is the reference angle for $\theta = -750^\circ$?

The angle $\theta = -750^\circ$ is two full rotations of 360° , plus an extra 30° in the negative direction, which means the angle is coterminal with $\theta = -30^\circ$. We want to convert this to a positive angle, which we can do by adding the negative angle to 360° .

$$\alpha = 360^\circ - 30^\circ$$

$$\alpha = 330^\circ$$

So the angle $\theta = -30^\circ$ is coterminal with $\alpha = 330^\circ$. That angle is in the fourth quadrant, which means the reference angle β is

$$\beta = 360^\circ - \alpha$$

$$\beta = 360^\circ - 330^\circ$$

$$\beta = 30^\circ$$

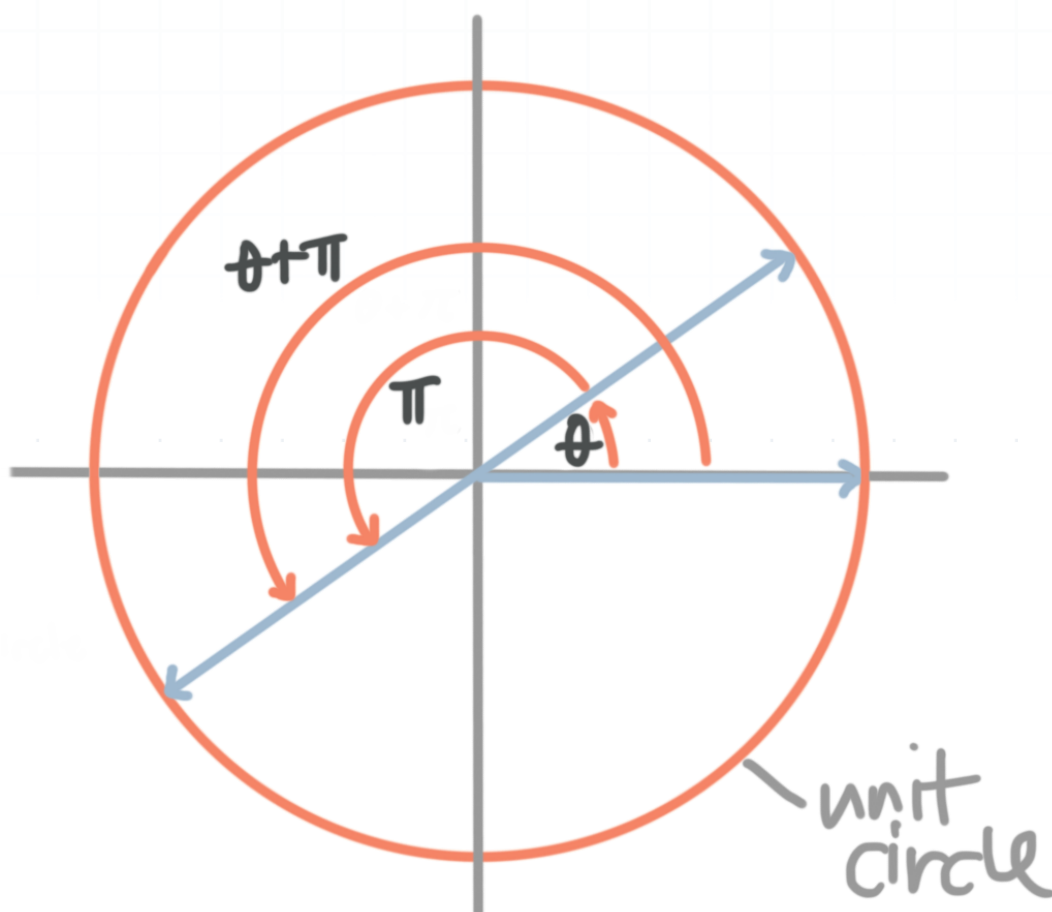


We'll do an example where we need to use symmetry to find sine and cosine values.

Example

If θ is an angle in the first quadrant such that $\cos \theta = 4/7$, use symmetry to find $\cos(\theta + \pi)$ and $\sin(\theta + \pi)$.

Since θ is in the first quadrant, $\theta + \pi$ is in the third quadrant.



By symmetry, we know that adding π radians to θ puts us at the same x -value, but negative, because we're in the third quadrant instead of the

first. In other words, adding π reflects the angle across the y -axis and then across the x -axis. So since $\cos \theta = 4/7$, we know that

$$\cos(\theta + \pi) = -\frac{4}{7}$$

To find $\sin \theta$, we'll substitute $\cos \theta = 4/7$ into the Pythagorean identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin^2 \theta + \left(\frac{4}{7}\right)^2 = 1$$

$$\sin^2 \theta + \frac{16}{49} = 1$$

$$\sin^2 \theta = 1 - \frac{16}{49}$$

$$\sin^2 \theta = \frac{49(1) - 16}{49}$$

$$\sin^2 \theta = \frac{33}{49}$$

$$\sin \theta = \pm \sqrt{\frac{33}{49}}$$

Since θ is in the first quadrant, $\sin \theta$ is positive, so

$$\sin \theta = \sqrt{\frac{33}{49}} = \frac{\sqrt{33}}{7}$$



By symmetry, we know that adding π radians to θ puts us at the same y -value, but in the third quadrant instead of the first. In other words, adding π reflects the angle across the y -axis and then across the x -axis. So

$$\sin(\theta + \pi) = -\frac{\sqrt{33}}{7}$$

Therefore, based on only the fact that $\cos \theta = 4/7$, and using a little symmetry, we know that

$$\cos(\theta + \pi) = -\frac{4}{7}$$

$$\sin(\theta + \pi) = -\frac{\sqrt{33}}{7}$$

Let's do another example.

Example

Let θ be an angle in the fourth quadrant such that $\sin \theta = -0.381$, and determine the values of $\sin(\theta - 3\pi)$ and $\cos(\theta - 3\pi)$.

First, we'll compute $\cos \theta$ by plugging $\sin \theta = -0.381$ into the Pythagorean identity.

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$(-0.381)^2 + \cos^2 \theta = 1$$



$$\cos^2 \theta = 1 - (-0.381)^2$$

$$\cos^2 \theta \approx 1 - 0.145$$

$$\cos^2 \theta \approx 0.855$$

$$\cos \theta \approx \pm \sqrt{0.855}$$

We've been given that θ is in the fourth quadrant, so $\cos \theta$ is positive and

$$\cos \theta \approx \sqrt{0.855}$$

$$\cos \theta \approx 0.925$$

Since θ is in the fourth quadrant, we can summarize what we know about the original angle. x is positive so $\cos \theta \approx 0.925$ and y is negative so $\sin \theta = -0.381$.

When we subtract 3π from the angle θ , that takes us one full rotation of -2π in the negative direction, and brings us back to the fourth quadrant, and then takes us another $-\pi$ rotation into the second quadrant. In the second quadrant, and by symmetry, our positive x -value becomes a negative x -value, and our negative y -value becomes a positive y -value. Therefore

$$\cos(\theta - 3\pi) \approx -0.925$$

$$\sin(\theta - 3\pi) = 0.381$$

